Two-Phase Flow in Hele-Shaw Models

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A study of the motion of the interface in two-phase flow in two dimensions with gravity force parallel to the plane of motion is described. The experiments were run in a porous medium analog, the Hele-Shaw model. A modification of the Saffman-Taylor solution is compared with the experimental results. The arbitrary parameter in the solution, measured in the experiments, does not approach an asymptote but is a function of the ratio of plate spacing to channel height. In the course of the air-water experiments three forms of the interface were observed before turbulence, which indicate regime changes in the flow. Finally, the effects of heterogeneities (change in plate spacing) on the interface were observed.

The unstable motion of two immiscible fluids in a porous medium is dependent on the properties of the fluids and the properties of the medium. The motion is unstable when a secondary fluid displaces a denser, in-place fluid from a porous medium (*11*). The motion is also unstable if a secondary fluid displaces a more viscous fluid (*3*, *9*, *12*). The effect of the surface tension of the fluids is to limit the range of perturbation for which the motion becomes unstable (*1*, *9*). The mathematical expression of the unstable interface for two-dimensional flow, where the gravity force is not in the plane of motion, contains an arbitrary parameter λ related to the width of the finger of secondary fluid (*6*, *8*, *9*). The value of λ has been observed to be asymptotic to 0.5 in a porous medium analog, the Hele-Shaw model (*9*). The mathematical solution is unique at this value of $\lambda = 0.5$ (*8*). The two-dimensional motion of the interface where the gravity force is parallel to the plane of motion is also of interest. This paper describes a study of two-phase flow in two dimensions with gravity force parallel to the plane of motion. We modified the solution of Saffman and Taylor (*9*) and compared interfaces calculated from this modified solution with experiments in Hele-Shaw models. The parameter λ in the modified equation is arbitrary and in the experiments did not approach an asymptote; λ is a function of the ratio of plate spacing b to channel height h. We found three different forms of the interface which indicate regime changes in the flow. Finally, the effects of heterogeneities (change in plate spacing) on the interface were observed in the flow experiments.

THEORY

The steady motion of a fluid in a porous medium is governed by Darcy's law, which states the flow velocity is proportional to the piezometer head divided by the viscosity of the fluid.

$$\mathrm{v} = -\frac{k}{\mu}\,\mathrm{grad}\,(p + \rho g y) \tag{1}$$

Two-dimensional flow in a porous medium may be studied analogously in a Hele-Shaw model. A Hele-Shaw model is constructed of two parallel plates that are close together, such that flow is between these two plates. The components of the average velocity in a Hele-Shaw model where gravity is in the plane of the flow are

$$v = -\frac{b^2}{12\mu}\left(\frac{\partial p}{\partial y} + \rho g\right) \tag{2}$$

$$u = -\frac{b^2}{12\mu}\,\frac{\partial p}{\partial x} \tag{3}$$

If

$$k = b^2/12 \tag{4}$$

there is a direct analogy between Darcy's law, Equation (1), and flow in a Hele-Shaw model, Equations (2) and (3). Also, the velocity potentials in a porous medium and in a Hele-Shaw model are given by Laplace's equation.

To study permeability variations in a porous medium analogously in a Hele-Shaw model, the plate separation of the model is changed. As long as plate separations are small, deviations from two-dimensional potential flow in heterogeneous Hele-Shaw models are either small (for small values of grad b) or are restricted to small areas (when grad b is not small) (*4*, *10*).

The equation of the interface for two-phase immiscible flow in a Hele-Shaw model with gravity not in the plane

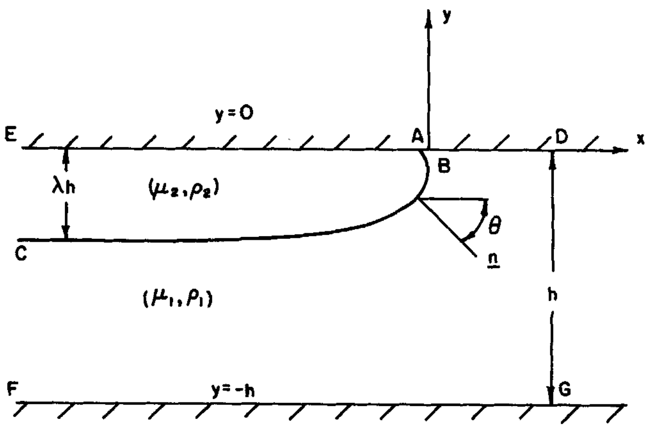

Fig. 1. Interface between two immiscible fluids.

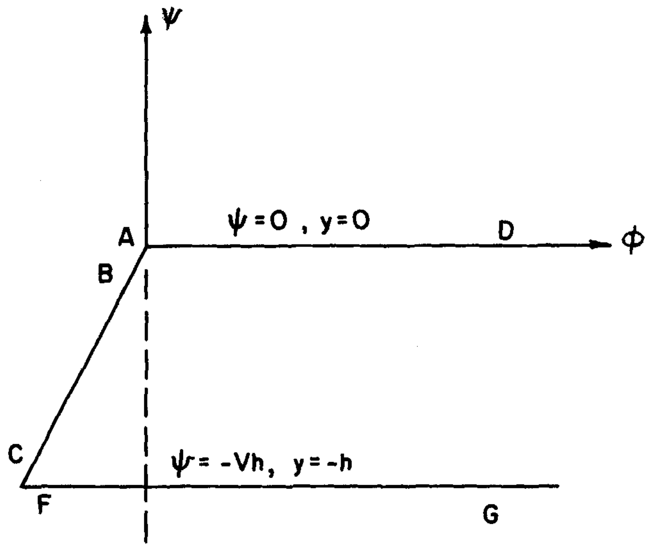

Fig. 2. Interface in potential plane.

of flow is a solution of Laplace's equation for the in-place fluid (9):

$$x = \frac{1-\lambda}{\pi} \ln 1/2 \left(1 + \cos \frac{\pi y}{\lambda}\right) \tag{5}$$

where λ is the ratio of the finger width to channel width and also by material balance considerations, the ratio of the average velocity of the primary fluid ahead of the interface V to the average velocity of the secondary fluid U. Thus

$$\lambda = V/U \tag{6}$$

Figure 1 is a sketch of the interface between two fluids moving in the positive x direction with gravity perpendicular to the bulk flow and in the plane of flow in a Hele-Shaw model (analogously a porous medium). The secondary fluid, subscript 2, moves with velocity U and the in-place fluid, subscript 1, moves with velocity V at $x = +\infty$. The boundaries of the model are at $y = 0$ and $y = -h$. The vertical height of the secondary fluid finger at $x = -\infty$ is $-\lambda h$. The velocity potential is

$$\phi_1 = -\frac{b^2}{12\mu}(p + \rho g y) \tag{7}$$

and the value of the streamline at $y = -h$ is $\psi_1 = -Vh$. In the potential plane of Figure 2 the boundary of the in-place fluid transforms onto a semi-infinite strip $\phi_1 > -GU\lambda h$ and $-Vh < \psi_1 < 0$, where

$$G = \frac{g\rho_1 b^2}{12\mu_1 U} \tag{8}$$

The solution of Laplace's equation is found in the same manner as for Equation (5), such that (7)

TABLE 1. PRIMARY FLUID PROPERTY VALUES

	Viscosity, centipoise		Density, g./cc.		Surface tension, g./sec.²	
	20°C.	27°C.	20°C.	27°C.	20°C.	27°C.
Glycerine*	415	311	1.25	1.23	63	63
Castor oil	674	571	0.96	0.95	40	40
Water	1.3	0.91	1.01	1.0	72	74

* Contained some water.

$$\psi_1 = \frac{Vh}{\lambda - 1} \sum_{n=1}^{\infty} A_n \exp\left\{-\frac{n\pi G\psi_1}{Vh}\right\} \sin \frac{n\pi\psi_1}{Vh} \tag{9}$$

The A_n's are found by collocation. This series was matched over the interval $-1 < \psi_1 \leq 0$. The linear algebraic equations obtained may be solved for the A_n's.* Equation (9) may be written in terms of the coordinates of the interface, such that

$$x = \frac{Gy}{\lambda} - h \sum_{n=1}^{\infty} A_n \exp\left\{-\frac{n\pi Gy}{\lambda h}\right\} \cos \frac{n\pi y}{\lambda h} \tag{10}$$

The solutions of Equations (5) and (10) neglect surface-tension forces at the interface of the two fluids. If

* This is not too easy since the matrix is ill conditioned. Solutions were only found for values of $G < 3$.

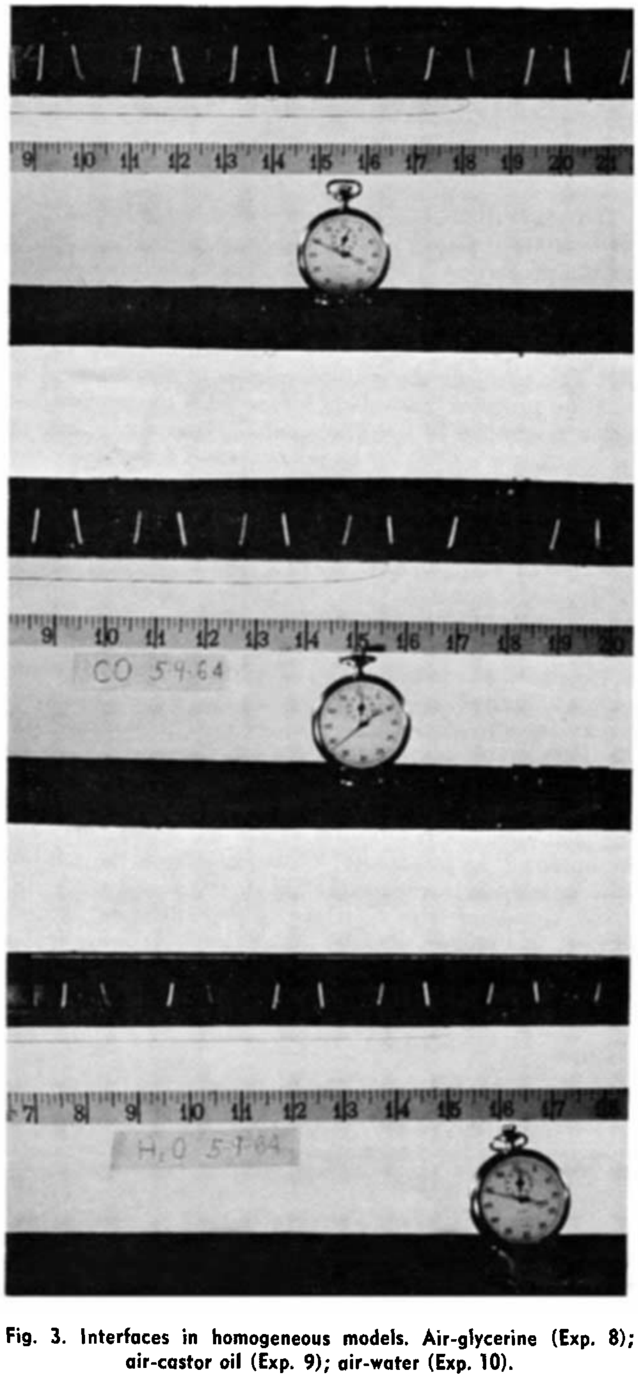

Fig. 3. Interfaces in homogeneous models. Air-glycerine (Exp. 8); air-castor oil (Exp. 9); air-water (Exp. 10).

we assume a potential which includes both the surface tension and the gravity force

$$p_2 = p_1 + \frac{2\sigma}{b} \tag{11}$$

where σ is the surface tension of the in-place fluid, then the potential becomes

$$\tilde{\phi}_1 = -\frac{\sigma b}{6\mu_1} + \frac{g\rho_1 b^2}{12\mu_1} y \tag{12}$$

If $\tilde{\phi}_1$ is used in the derivation of Equation (10), then the coordinates of the interface including surface-tension forces can be calculated. For $G < 3$ the surface-tension force is not important so we did not find a numerical solution of Equation (10) that included surface tension.

Since Equation (10) is parametric we want to predict the dependence of the parameter λ on the properties of the model or the porous medium. An asymptotic value of λ was never reached in our experiments nor will it necessarily be reached in application of Equation (5) to flow in a real porous medium. Dimensional analysis of the variables and parameters in the problem

$$f(x, y, U, V, b, h, p, g, \sigma, \rho, \mu) = 0 \tag{13}$$

yields a functional equation with a possible set of dimensionless groups:

$$F\left(x/h, y/h, \frac{V}{U}, \frac{b}{h}, \frac{p_2}{\rho V^2}, \frac{g\rho b^2}{\mu U}, \frac{\sigma}{\mu U}, \frac{hV\rho}{\mu} \right) = 0 \tag{14}$$

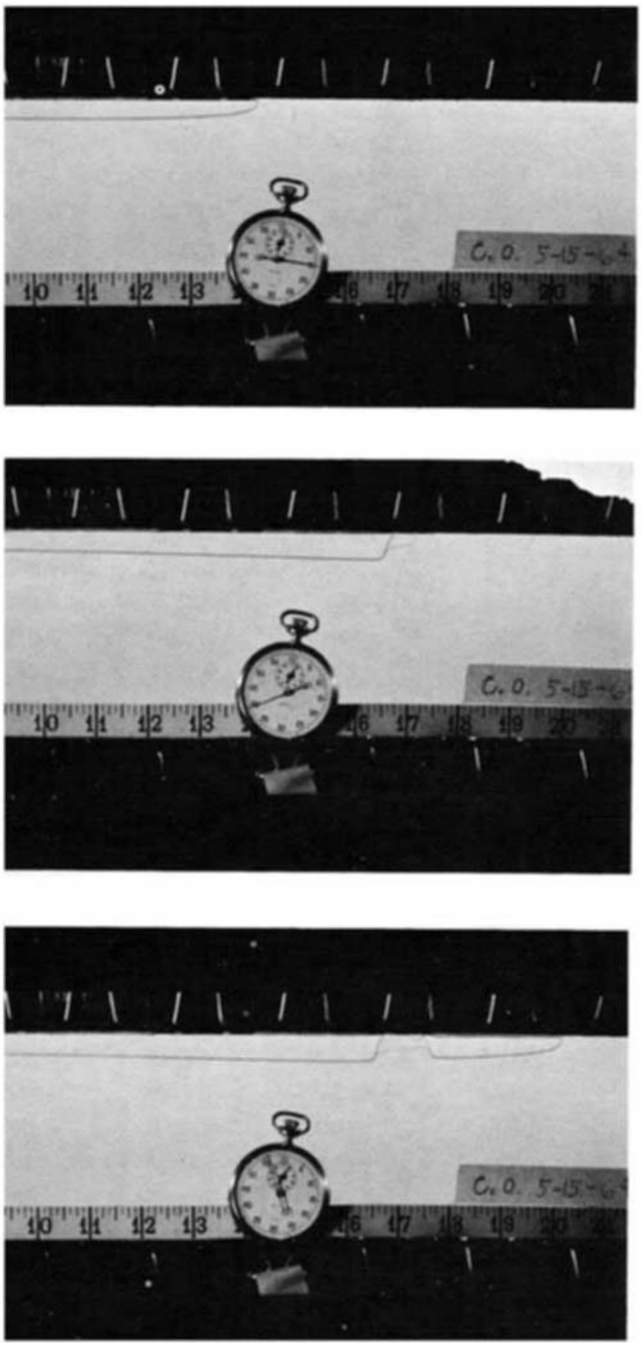

Fig. 4. Interface in partially heterogeneous model. Air-castor oil (Exp. 21).

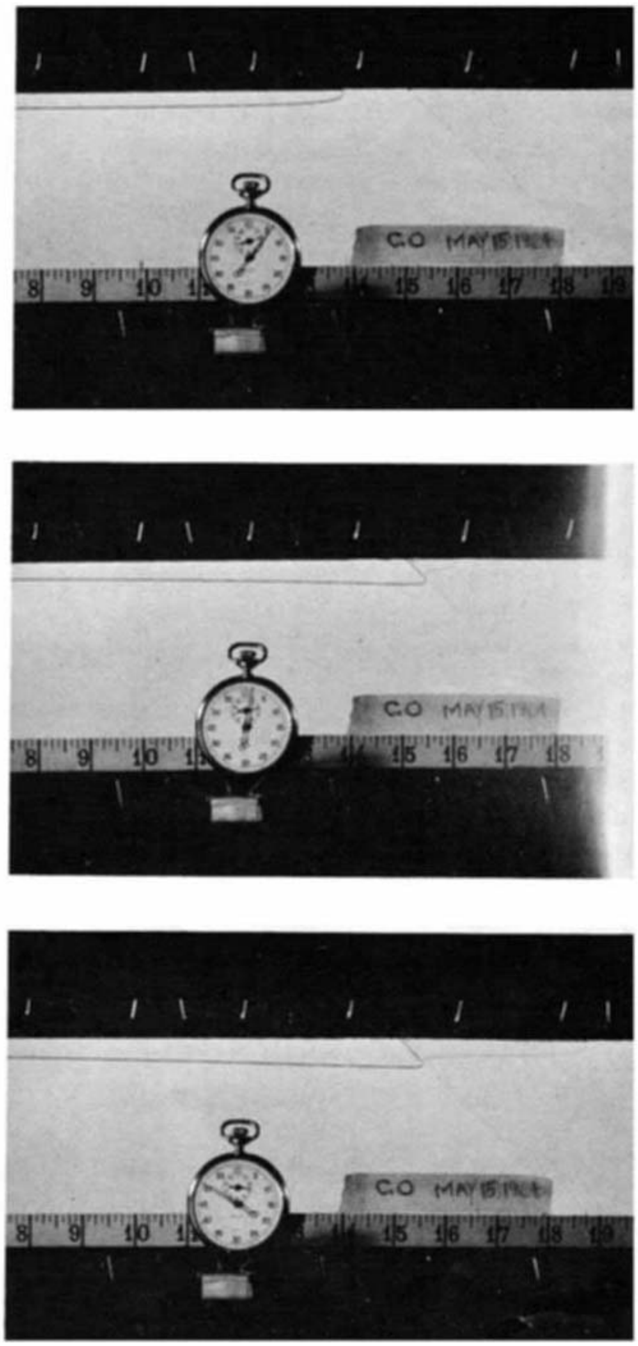

Fig. 5. Interface in completely heterogeneous model. Air-castor oil (Exp. 24).

Table 2. Experimental Summary

Exp. No.	Primary fluid	b, cm.	h, cm.	U, cm./sec.	G	T	R	λ
1	Water	0.180	10.18	0.80	369.00	29.300	7.2400	0.045
2	Glycerine	0.180	10.18	1.59	0.49	0.028	0.0038	0.063
3	Glycerine	0.180	10.18	0.40	1.98	0.110	0.0053	0.058
4*	Glycerine	0.180	10.18	0.88	0.89	0.051	0.0120	0.100
5	Glycerine	0.128	10.18	0.26	1.50	0.120	0.0003	0.110
6	Glycerine	0.128	10.18	0.35	1.14	0.091	0.0004	0.142
7*	Glycerine	0.128	10.18	0.30	1.34	0.110	0.0004	0.142
8	Glycerine	0.152	10.18	0.67	0.94	0.063	0.0013	0.098
9	Castor oil	0.152	10.18	0.43	0.61	0.034	0.0004	0.108
10	Water	0.157	10.18	1.56	135.00	12.700	1.0100	0.075
11	Glycerine	0.157	10.18	0.67	1.13	0.074	0.0016	0.091
12	Glycerine	0.157	10.18	0.14	5.37	0.350	0.0003	0.056
13	Castor oil	0.157	10.18	0.43	0.66	0.035	0.0004	0.098
14	Water	0.157	10.18	0.45	563.00	51.200	0.3400	0.072
15	Water	0.157	10.18	0.25	1,025.00	93.300	0.1800	0.075
16	Castor oil	0.157	10.18	0.22	1.31	0.070	0.0002	0.070
17	Glycerine	0.155	10.18	1.06	0.83	0.054	0.0028	0.134
18	Castor oil	0.155	10.18	0.93	0.30	0.017	0.0008	0.173
19	Water	0.170	10.18	4.44	59.70	5.020	3.5600	0.093
20	Glycerine	0.170	10.18	0.86	1.22	0.072	0.0027	0.135
21	Castor oil	0.170	10.18	0.76	0.45	0.022	0.0008	0.135
22	Water	0.170	10.18	3.50	75.70	6.360	2.8000	0.072
23	Glycerine	0.170	10.18	0.27	3.96	0.240	0.0008	0.064
24	Castor oil	0.170	10.18	0.25	1.38	0.068	0.0003	0.071
25	Castor oil	0.094	10.18	0.26	0.40	0.035	0.0001	0.152
26	Water	0.155	10.18	5.60	27.00	2.580	2.6000	0.072
27	Water	0.155	10.18	9.02	16.70	1.600	4.1900	0.092
28	Water	0.155	10.18	51.00	2.96	0.280	23.7200	0.184
29	Water	0.170	10.18	15.25	11.91	1.040	8.3800	0.079
30	Water	0.170	10.18	16.68	10.89	0.950	9.1600	0.082
31	Water	0.170	10.18	32.20	5.64	0.490	17.6800	0.092
32	Water	0.170	10.18	29.20	6.22	0.540	16.0400	0.189
33	Water	0.170	10.18	8.80	22.37	1.950	5.1100	0.082
34	Water	0.090	10.18	10.61	5.20	0.860	1.8000	0.088
35	Water	0.170	10.18	11.90	16.54	1.440	6.9100	0.080
36	Water	0.090	10.18	9.11	6.06	1.000	1.5400	0.101
37	Water	0.170	10.18	23.40	8.41	0.730	13.5800	0.090
38	Water	0.090	10.18	15.82	3.49	0.570	2.6800	0.091
39	Water	0.170	10.18	27.30	7.21	0.630	15.8500	0.122
40	Water	0.170	10.18	24.20	8.13	0.710	14.0500	0.138
41	Water	0.175	5.09	6.90	33.00	5.440	6.3100	0.122
42	Water	0.170	10.18	1.20	180.00	15.250	0.7700	0.063
43	Water	0.165	20.36	2.86	70.70	3.090	0.8800	0.032
44	Glycerine	0.175	5.09	0.55	1.82	0.210	0.0034	0.160
45	Glycerine	0.170	10.18	0.94	1.01	0.060	0.0027	0.092
46	Glycerine	0.165	20.36	0.94	0.96	0.029	0.0013	0.077
47	Castor oil	0.175	5.09	0.49	0.87	0.083	0.0013	0.195
48	Castor oil	0.170	10.18	0.72	0.56	0.027	0.0028	0.161
49	Castor oil	0.165	20.36	1.01	0.37	0.009	0.0058	0.117

* 5-deg. slope.

For creeping flow the inertial terms are small and we assume that the surface-tension forces are negligible. At $\lambda = V/U = y/h$, the group x/h is constant and Equation (14) becomes

$$F\left(\lambda, \frac{b}{h}\right) = 0 \tag{15}$$

and we might expect λ to correlate with b/h for the model ($\sqrt{k}/h$ for a porous medium).

EXPERIMENT

The theoretical study provides a solution to the two-dimensional equations of motion for a limited range of the value of G. This solution contains an arbitrary parameter λ that may be related to the plate spacing of the model or permeability of a porous medium. The objective of the experimental program was to provide a check of the mathematical solution and a check of the validity of the value of $\lambda = V/U$. Also we obtained data to correlate λ as a function of b/h. If surface tension is important it may show up in experiments either as a discrepancy between theory and experiment or as a regime change in the fluid flow. Finally, we observed the phenomenological effects of heterogeneity (variation in plate spacing in the model or permeability in the medium) on the flow.

The experiments were run in Hele-Shaw models made from two ¼-in. plates of glass usually 4 to 6 ft. long. The channel heights were 2, 4, and 8 in. The plates were separated by rubber strips 0.128 to 0.180 in. thick. The plates were held together by larger paper clips, fitted with a plastic header at one end for injection of fluid, and mounted in a wooden stand. Some of the models were made heterogeneous by gluing celluloid onto one of the plates. One type of heterogeneity was a partial ellipse in the center of the model; the second was a space change over one-half the length of the model. Air was always used as the secondary fluid to drive out glycerine, castor oil, or water. Air under constant pressure was blown in through the header, forcing the in-place fluid out the other end of the model. The displaced fluid ran over a dam at the outlet such that the outlet pressure was maintained at the head of in-place fluid. Input pressures were measured by a water manometer. A tape measure and a stop watch were attached to the front of the model and results of the experiments were recorded with still and motion pictures. The velocity and shape of the interface were determined from the photographs. In some of the experiments a vertical line of dye was placed in the model with a hypodermic needle to provide a measure of the in-place fluid velocity. A Cannon-Fenske-Ostwald viscometer was used to determine the viscosity of the fluids; a ring tensiometer was used to determine the surface tension of the fluids; a pycnometer was used for density. The property values are summarized in Table 1. Photographs of typical experiments for the three in-place fluids displaced with air are shown for a homogeneous model in Figure 3, for a partial heterogeneity in Figure 4, and for the complete heterogeneity in Figure 5. The results of a total of forty-nine experiments are summarized in Table 2.

DISCUSSION

Figure 6 shows the comparison of the interface from experiments 2 and 16 (air-glycerine and air-castor oil) with the interface calculated from Equation (10). If the value of λ is measured from an experiment and used in Equation (10) the calculated interface (closed circles) does not match the experiment. However, the experiment can be matched by varying λ (open circles). It is assumed in the solution of Equation (10) that the in-place fluid moves in plug flow and that $\lambda = V/U$. If $V = V(y)$, then $\lambda = \lambda(y)$ and we wouldn't expect the measured λ to provide the correct solution. However, experiments run with a vertical line in place show that the in-place fluid moves in plug flow and further $\lambda = V/U$. Figure 7 is a tracing of an air-castor oil experiment (experiment 48) which shows the movement of the vertical line relative to the fluid interface. The results of nine experiments, three for each in-place fluid, show that $\lambda = V/U$ within experimental error. The results of these experiments, corrected for dispersion, are in Table 3.

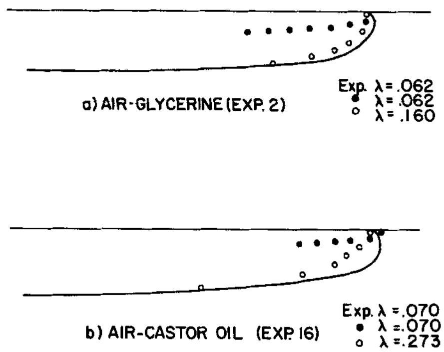

Fig. 6. Comparison of experiment with calculation, Equation (10).

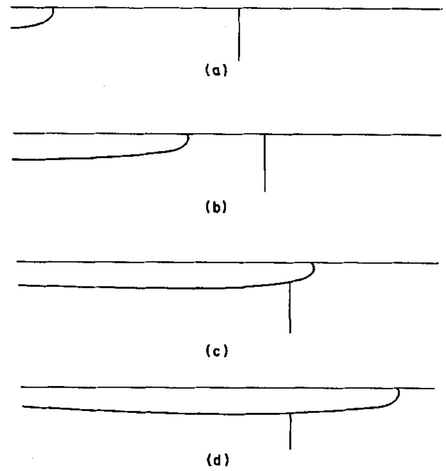

Fig. 7. Interface with dye line in primary fluid. Air-castor oil (Exp. 48).

Dimensional analysis of the variables indicated λ to be a function of b/h. This is true for the experiments shown in Figure 8 where λ is plotted vs. b/h (experiments 41-49). Figure 8 also shows that λ does not approach an asymptote in this range of velocities and b/h ratios. It is probable that the relative position and slope of the three lines are related to gravity and surface-tension forces.

To determine if the surface-tension forces are important, we did order-of-magnitude calculations for the experiments which contained the vertical line (measured in-place fluid velocity). The capillary pressure due to surface tension of the fluid opposes the motion of the interface which slows the interface, so it will be wider for a given overall pressure drop. In Table 4 the velocity of the in-place fluid is

TABLE 3. COMPARISON OF λ AND V/U

Exp. No.	Primary fluid	λ	0.67 (V/U)
41	Water	0.12	0.14
42	Water	0.06	0.06
43	Water	0.03	0.02
44	Glycerine	0.16	0.12
45	Glycerine	0.09	0.12
46	Glycerine	0.08	0.06
47	Castor oil	0.20	0.32
48	Castor oil	0.16	0.12
49	Castor oil	0.12	0.05

TABLE 4. VELOCITY COMPARISON

Exp. No.	Primary fluid	V, cm./sec. from λ	V, cm./sec. from ΔP	V, cm./sec from T
41	Water	0.08	4.26	—3.0
42	Water	0.08	4.53	—3.1
43	Water	0.09	4.80	—3.2
44	Glycerine	0.09	0.10	—0.01
45	Glycerine	0.09	0.10	—0.01
46	Glycerine	0.07	0.11	—0.01
47	Castor oil	0.09	0.06	—0.004
48	Castor oil	0.12	0.07	—0.004
49	Castor oil	0.12	0.07	—0.004

given for nine experiments calculated from the measured height of the interface and the secondary fluid velocity. Also, the values of the in-place fluid were estimated from the pressure drop across the model. Finally the velocity of the in-place fluid due to surface-tension was estimated. Table 4 shows the surface-tension force is important for the air-water experiments.

For the air-water experiments three forms of the interface were observed before distinct turbulence. These three

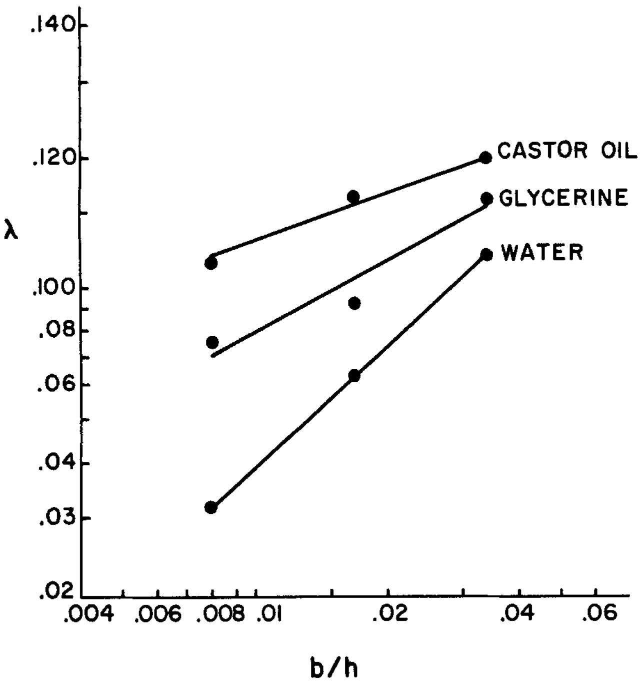

Fig. 8. λ vs. b/h.

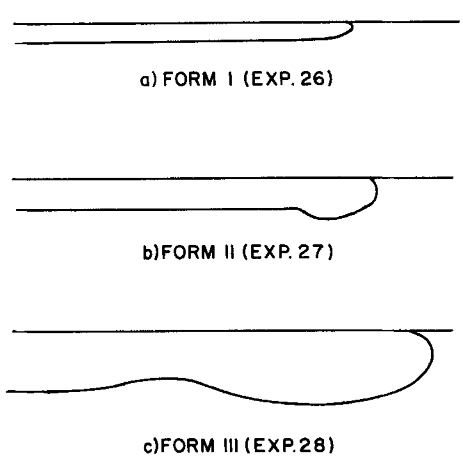

Fig. 9. Three forms of air-water interface.

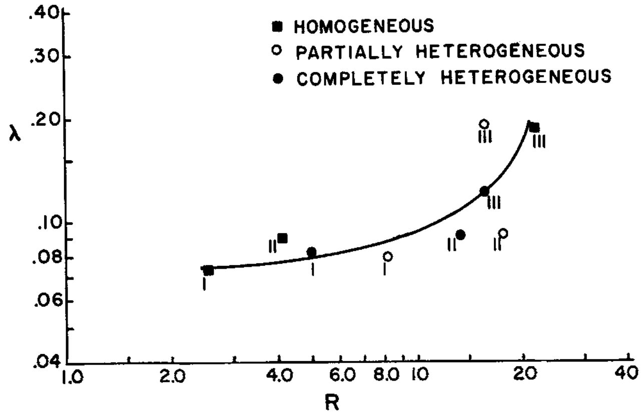

Fig. 10. λ vs. R; air water.

forms are shown in Figure 9. A plot of λ for these experiments vs. Reynolds number in Figure 10 shows that for the final form the regime changes to an inertial regime. Since the first form is predominantly a viscous regime, and the results in Table 4 indicate that surface tension may be important, we assume that the second form is predominantly a surface-tension regime. Although flow in a Hele-Shaw model is analogous to flow in a porous medium, for $R > 1$ the analogy may break down for the final form. We plan further experiments in actual porous media models to check these interface shapes.

The results of placing heterogeneities in the model are that the finger of secondary fluid moves proportional to the average resistance of the model; the interfaces do not appear to be different except when the interface passes a heterogeneous boundary. The heterogeneities cause shifts in interface form in the air-water experiments.* Figures 3, 4, and 5 are photographs of these experiments.

CONCLUSIONS

We conducted a study of two-phase immiscible flow in a porous medium analog, the Hele-Shaw model, with gravity in the plane of flow. A modification of the solution of Saffman and Taylor, which includes gravity, compares with the experiments where surface-tension force is not important. The solution is parametric, but dimensional analysis indicates and experiments show that the parameter λ is a function of the ratio of plate spacing to channel height (permeability to cross-sectional area in a porous medium). Heterogeneities in the flow field modify the width of the interface such that for a complete heterogeneity the width is determined by an average model resistance. Three forms of the interface were observed with the air-water system that may be due to change from viscous to surface tension to inertial regimes. Heterogeneities in the model modify these forms such that for high resistance the form changes to the one apparently dominated by viscous forces.

ACKNOWLEDGMENT

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NOTATION

A = cross-sectional area, sq.cm.
b = plate spacing, cm.

* Movies of the experiments that show the phenomenological effect of the heterogeneities are available.

G = dimensionless gravity, $gb^2\rho/12\mu U$
g = acceleration due to gravity, cm./sec.^{2}
h = channel height, cm.
k = permeability, sq.cm.
n = integer (dimensionless)
p = pressure, g./sq.cm.
R = dimensionless velocity, $b^2U\rho/\mu h$
T = dimensionless surface tension, $\sigma b/6\mu Uh$
U = secondary fluid velocity, cm./sec.
u = average velocity in x direction, cm./sec.
V = primary fluid velocity, cm./sec.
v = average velocity in y direction, cm./sec.
$\mathbf{v}$ = velocity vector
x = coordinate, cm.
y = coordinate, cm.

Greek Letters

λ = finger width, $y/h \equiv V/U$
ϕ = flow potential, sq.cm./sec.
ψ = stream function, sq.cm./sec.
ρ = density, g./cc.
σ = surface tension, g./sec.^{2}
μ = viscosity, centipoise

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Stability of Loop Reactors

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A method of determining all the possible steady states of a tubular recycle reactor with plug flow is presented. The information needed to determine the steady states can be used to ascertain the asymptotic and global stability of the reactor without performing transient computations. The effects of changes in the recycle ratio and start-up conditions are examined. Several numerical examples are included to demonstrate the use of the method for various types of recycle reactors.

Many industrial reactors use recycling as a convenient means of temperature and concentration control, or to improve the yield if conversion per pass is small. An extensive survey of chemical processes that use recycle can be found in the monograph by Nagiev (*1*).

Few papers have investigated the stability problems associated with the operation of such reactors. It has been shown (2) that under certain conditions the use of a recycle stream in a tubular reactor might give rise to instabilities. That treatment was, however, limited to a case in which only one steady state could exist and the criteria which were derived were based upon the tedious numerical computations of a matrizant. Gall and Aris (*3*) considered the problem of steady state operation of a tubular reactor followed by a stirred tank with mass and thermal feedback. It was demonstrated that the presence of feedback affects considerably the heat generation curve of the stirred tank. Reilly and Schmitz (*6*) have recently developed a method of analyzing the stability of a loop reactor from numerical solutions of the steady state equations. This paper presents a new and different approach to that problem.

In many processes the recycle system consists of a reactor connected to a separation unit such as a distillation column. The behavior of the combined system is largely dependent on the operation and characteristics of the specific separation unit, which often masks the inherent pathological characteristics of the reactor. The aim of this work is to investigate the special trends which are caused by a recycle stream. The discussion is therefore limited to loop reactors; that is, a reactor in which there is no separation unit and no lag in time between the exit and the reentrance of the recycle stream to the reactor. The following cases will be considered: recycle in a tubular loop adiabatic reactor; recycle in a semi-isothermal loop reactor with constant feed concentration; and recycle in a semi-isothermal loop reactor.